

On the Stochastic Rank of Metric Functions

Nikolay Balov

April 24, 2009

Abstract

For a class of integral operators with kernels metric functions on manifold we find some necessary and sufficient conditions to have finite rank. The problem we pose has a stochastic nature and boils down to the following alternative question. For a random sample of discrete points, what will be the probability the symmetric matrix of pairwise distances to have full rank? When the metric is an analytic function, the question finds full and satisfactory answer. As an important application, we consider a class of tensor systems of equations formulating the problem of recovering a manifold distribution from its covariance field and solve this problem for representing manifolds such as Euclidean space and unit sphere.

1 Problem formulation and motivation

We start with the classical Fredholm integral equation of the first kind

$$\int_V \psi(x, y) f(y) dy = g(x), \quad x \in U, \quad (1)$$

where U and V are open sets in \mathbb{R}^n and $f : U \rightarrow \mathbb{R}$, $g : V \rightarrow \mathbb{R}$, and $\psi : U \times V \rightarrow \mathbb{R}$ are some functions. Depending on the domains more conditions on f , g and ψ may be necessary for the correct formulation of (1). Let $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ be two discrete samples of points chosen by uniform distributions on U and V respectively. Then equation (1) can be discretized by the matrix-quadrature method

$$\sum_{j=1}^k \psi(x_i, y_j) f(y_j) = g(x_i), \quad i = 1, \dots, k. \quad (2)$$

In general, the inverse problem of solving (1) for f , is often ill-posed and approximation based on (2) will eventually result in increasingly unstable solution as k increases. Here, however, we are interested in the first potential obstacle to solve (2) - the matrix $\Psi := \{\psi(x_i, y_j)\}_{i,j=1,1}^{k,k}$ may not be of full rank. The problem is stochastic one for the points x_i 's and y_j 's are chosen in random fashion. In section 2 we investigated it and find some conditions for the kernel ψ , which guarantee that for any k , Ψ has full rank with probability one. We also show some necessary and sufficient conditions for analytic kernels ψ to be of finite rank.

A further generalization of equation (1) takes f to be a function on n -manifold M and g and ψ to be linear operator fields on M . For a point $p \in M$ with M_p we denote the tangent space at p and with $T_1^1(M_p)$, the vector space of (1,1)-tensors (linear operators) on M_p . Let μ be a measure on M as for example the volume measure $V(p)$ on Riemannian manifolds. Consider the equation

$$\int_M Y(p, q) f(q) d\mu(q) = C(p), \quad p \in M, \quad (3)$$

such that $Y(p, \cdot) \in T_1^1(M_p)$ and $C(p) \in T_1^1(M_p)$. The inverse problem here is finding f for given fields Y and C . If we know that (3) has a unique solution for f then it can be found by solving

$$\int_M \text{tr}(Y(p, q)) f(q) d\mu(q) = \text{tr}(C(p)), \quad (4)$$

an equation of type (1).

The importance of the class (3) of tensor equations is that it contains the problem of recovering a distribution from its covariance field. Next, we briefly pose this problem, while more details one can find in [1].

Let M be a Riemannian manifold with metric tensor G . For any $p \in M$, $G(p) \in T^2(M_p)$ is a co-variant 2-tensor. Let $\text{Exp}_p : M_p \rightarrow M$ be the exponential map at p and $\mathcal{U}(p) \subset M$ be the maximal normal neighborhood of p , where Exp_p^{-1} is well defined. Note that since $\text{Exp}_p^{-1}q \in M_p$, $(\text{Exp}_p^{-1}q)(\text{Exp}_p^{-1}q)^T \in T_2(M_p)$, a contra-variant 2-tensor. For a density function $f \geq 0$ on M , the covariance operator field of f is $G\Sigma : M \rightarrow T_1^1(M)$, such that for any $p \in M$

$$G\Sigma(p) := \int_{\mathcal{U}(p)} G(p)(\text{Exp}_p^{-1}q)(\text{Exp}_p^{-1}q)^T f(p) dV(p). \quad (5)$$

The problem of distribution recovering is of type (3) if we take $\mu = V$, $C = G\Sigma$ and $Y(p, q) = G(p)(\text{Exp}_p^{-1}q)(\text{Exp}_p^{-1}q)^T$. Note that

$$\text{tr}(Y(p, q)) = \text{tr}(G(p)(\text{Exp}_p^{-1}q)(\text{Exp}_p^{-1}q)^T) = d^2(p, q),$$

is the square geodesic distance on M . Thus equation (4), specifically, is

$$\int_{\mathcal{U}(p)} d^2(p, q)f(q)dV(q) = g(p). \quad (6)$$

In the context of problem (6) we are interested in finding the rank of an integral operator of the form $L_\psi : f \mapsto \int \psi(p, q)f(q)dV(q)$, where $\psi(p, q) = d^2(p, q)$ is a square distance function on M . In particular, in section 3 we study the rank of the Euclidean metric $d(p, q) = \|p - q\|$ and the rank of the standard metric on the unit sphere \mathbb{S}^n , $d(p, q) = \cos^{-1}(\langle p, q \rangle)$, and show that while the Euclidean metric is of finite rank, the metric on the sphere is not. The last fact can be re-phrased as follows. For any discrete sample of points on a sphere, the square matrix of pairwise distances is non-singular with probability one.

The problem of establishing the non-singularity of the kernel of operator (1) is important in the context of more general statistical inverse problems on manifolds, as considered in [4], [11], [12] and [14]. Let $g(p) = f(p) + \epsilon$, where ϵ is a mean zero random variable with small variance, be a model with unknown regression function f . For a kernel ψ , the inverse problem with random noise is formulated as estimation of $L_\psi f$ from observations (p_i, g_i) . In [4], Cavalier and Tsybakov estimate f from the model $g = L_\psi f + \epsilon$, which is a noised version of (1). Usually points p_i are assumed uniformly distributed on M . If the kernel ψ has a finite rank, then these problems are necessarily ill-posed, since the operator L_ψ is not invertible.

Statistical inverse estimation is addressed in details in [12], [14], [4], [15] and [11]. In addition, in [2] different alternatives for regularization of ill-posed linear problems are discussed. In our final remarks we also briefly elaborate on this important aspect of all inverse problems. However, instead of regularizing a fixed kernel, we alleviate the ill-conditioning by choosing a kernel with better conditional number from a class of similar kernels. This is a completely different kind of approach. At the end, we show some simulation results supporting our proposal.

It is a common assumption for L_ψ to be Hermitian and even compact operator, although Mair and Ruymgaart, [13], and Cavalier, [3], relax the

assumptions to Hermitian and positive definiteness. There are symmetric operators however, as the covariance one shown above, for which even these assumptions are too strong and this makes relevant the subject of this study - the rank of a general operator with analytic kernel.

Remark 1 *Provided that there is a local chart (\tilde{U}, ϕ) , $\tilde{U} \subset \mathbb{R}^n$ on M such that $\phi(\tilde{U})$ contains the support of f , Y and C , equations (3), (5) and (4) can be formulated as integral equations in Euclidean space \mathbb{R}^n by substituting p and q with $\phi(x)$ and $\phi(y)$ respectively. In fact, the above requirement is not a strong one for complete Riemannian manifolds where $V(M \setminus \mathcal{U}(p)) = 0$. That is why, for the sake of simplicity, to the end of this exposition we will work in Euclidean settings.*

2 Rank of bi-variate analytic functions

Our goal here is to find some necessary and sufficient conditions for an integral kernel to be of finite rank. We restrict to real analytical kernels, i.e. bi-variate real analytical functions. The exposition uses elementary functional analysis techniques with some textbook facts presented for the sake of consistency.

Let U be an open subset of \mathbb{R}^n . A collection of functions $f_s : U \rightarrow \mathbb{R}$ is said to be linear independent in U if $\sum_s \alpha_s f_s(x) = 0$, for almost all (by Lebesgue measure) $x \in U$ only if $\alpha_s = 0$ for all s .

Lemma 1 *If functions $f_1(x), \dots, f_k(x)$ are linearly independent in U , then there exist x_1, \dots, x_k , such that $\text{rank}(\{f_i(x_j)\}_{i=1,j=1}^{k,k}) = k$.*

Proof. Since $f_1(x)$ is not identically zero, there is $x_1 \in U$ such that $f_1(x_1) \neq 0$. Determinant

$$\det \begin{pmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x) & f_2(x) \end{pmatrix} = f_1(x_1)f_2(x) - f_2(x_1)f_1(x)$$

can not vanish for all $x \in U$ or f_1 and f_2 would be linear dependent. Therefore, there is x_2 such that $\det(\{f_i(x_j)\}_{i=1,j=1}^{2,2}) \neq 0$. The selection process can be extended to find x_1, \dots, x_k in U , such that $\det(\{f_i(x_j)\}_{i=1,j=1}^{k,k}) \neq 0$. At the last step, defining $A_1 := \det(\{f_i(x_j)\}_{i=1,j=1}^{k-1,k-1}) \neq 0$, we choose $x_k \in U$ such

that for $x = x_k$

$$\det \begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots & f_k(x_1) \\ \dots & \dots & \dots & \dots \\ f_1(x_{k-1}) & f_2(x_{k-1}) & \dots & f_k(x_{k-1}) \\ f_1(x) & f_2(x) & \dots & f_k(x) \end{pmatrix} = \sum_{i=1}^k A_i f_i(x) \neq 0.$$

□

Definition 1 We say that function $\psi : U \times V \rightarrow \mathbb{R}$, $U, V \subset \mathbb{R}^n$ has a rank k and write $\text{rank}(\psi) = k$ if for any $m \in \mathbb{N}$, $x_i \in U$ and $y_i \in V$, $i, j=1, \dots, m$,

$$\text{rank}(\{\psi(x_i, y_j)\}_{i=1, j=1}^{m, m}) \leq k,$$

and k is the smallest number with this property.

Let $\psi : U \times V \rightarrow \mathbb{R}$, be a smooth bi-variate function, a fact we denote with $\psi \in C^\infty(U \times V)$. We say that ψ is analytic in V about a point $p \in V$ if ψ has Tailor expansion in $y \in V$

$$\psi(x, y) = \sum_{m=0}^{\infty} \sum_{s: \sum_{i=1}^n s_i = m} c_{s_1 \dots s_n}(x) (y_1 - p_1)^{s_1} \dots (y_n - p_n)^{s_n},$$

which for the sake of brevity we will write as

$$\psi(x, y) = \sum_{[s]=0}^{\infty} c_s(x) (y - p)^s,$$

where $s = (s_1, \dots, s_n)$ is a multi-index and $[s] = s_1 + \dots + s_n$.

We say that the space spanned by the functions c_s have a finite basis if there exist a number m and functions f_1, \dots, f_m , such that c_s 's are linear combinations of f_l 's, i.e. $c_s \in \text{span}\{f_1, \dots, f_m\}$ for all s .

Next result gives some necessary and sufficient conditions for ψ to have a finite rank.

Proposition 1 For a function $\psi \in C^\infty(U \times V)$ that is analytic in V about a point $p \in V$ the following three conditions are equivalent

(1) $\text{rank}(\psi) = k$.

- (2) $\psi(x, y) = \sum_{j=1}^k f_j(x)g_j(y)$ for linearly independent functions $f_j \in C^\infty(U)$ and $g_j \in C^\infty(V)$, $j=1, \dots, k$.
- (3) There are linearly independent functions $f_j \in C^\infty(U)$, $j=1, \dots, k$, such that all Taylor functions c_s are their linear combinations in U , i.e. $c_s \in \text{span}\{f_1, \dots, f_k\}$.

Proof. Without loss of generality we assume that p is the origin 0. We show that (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3).

Let condition (3) hold. Then for any s , $c_s(x) = \sum_{i=1}^k \beta_s^i f_i(x)$ and $\psi(x, y) = \sum_{i=1}^k (\sum_s \beta_s^i y^s) f_i(x)$, provided that for any i and y , $\sum_s \beta_s^i y^s$ converges. Suppose the contrary, that there exists i_0 and $y_0 \in V$ such that $\sum_s \beta_s^{i_0} y_0^s$ does not converge. Choose $x_j \in U$, $j=1, \dots, k$, such that $\text{rank}(A := \{f_i(x_j)\}_{j=1, i=1}^{k, k}) = k$. Let $\|A^{-1}\| > 0$ denote the norm of matrix A^{-1} . Fix a number $\epsilon > 0$. Since all $\sum_s c_s(x_j) y_0^s$ converge, there is N such that

$$\left| \sum_{[s]=N}^M \left(\sum_{i=1}^k \beta_s^i f_i(x_j) \right) y_0^s \right| < \epsilon / \|A^{-1}\|,$$

for any j and $M \geq N$. Also by the assumption, there is $M > N$, such that

$$\left| \sum_{[s]=N}^M \beta_s^{i_0} y_0^s \right| \geq \epsilon.$$

Define k -vectors $z = (z_1, \dots, z_k)^T$ and $w = (w_1, \dots, w_k)^T$, where

$$z_i := \sum_{[s]=N}^M \beta_s^i y_0^s \text{ and } w_j := \sum_{[s]=N}^M \left(\sum_{i=1}^k \beta_s^i f_i(x_j) \right) y_0^s.$$

The system $Az = w$ can be solved for z , $z = A^{-1}w$. Then $\|z\| \leq \|A^{-1}\| \|w\| < \epsilon$, which contradicts $\|z\| \geq |z_{i_0}| \geq \epsilon$. Therefore, the initial assumption is false and

$$g_i(y) := \sum_s \beta_s^i y^s,$$

are well defined functions in U . Thus condition (2) holds.

That (1) follows from (2) is obvious. Indeed, for $x_i \in U$, $y_i \in V$, $i=1, \dots, k$, define k -vectors $v_j = (g_j(y_1), \dots, g_j(y_k))^T$ and $\psi_i = (\psi(x_i, y_1), \dots, \psi(x_i, y_k))^T$.

Then for the column-vectors of matrix $\{\Psi = \psi(x_i, y_j)\}_{i,j=1}^{k,k}$ we have $\psi_i = \sum_{j=1}^k f_j(x_i)v_j$ which shows $\text{rank}(\Psi) = k$.

Next we show $(1) \Rightarrow (3)$. Let $\text{rank}(\psi) = k$. For fixed $x_i \in U$, $i=1, \dots, k+1$, consider the set

$$\psi(x_1, y), \dots, \psi(x_{k+1}, y)$$

of functions of y . Since no full rank matrix $\{\psi(x_i, y_j)\}_{i=1, j=1}^{k+1, k+1}$ exists, by lemma 1, $\{\psi(x_i, y)\}_{i=1}^{k+1}$ are not linear independent. Thus there are functions $\alpha_i(x)$, $x = (x_1, \dots, x_{k+1})$, not all zero, such that

$$\sum_{i=1}^{k+1} \alpha_i(x) \psi(x_i, y) = 0, \forall y \in V.$$

By expanding ψ in y we obtain

$$\sum_s \sum_{i=1}^{k+1} \alpha_i(x) c_s(x_i) y^s = 0, \forall y \in V,$$

equivalent to

$$\sum_{i=1}^{k+1} \alpha_i(x) c_s(x_i) = 0, \forall s,$$

and this is true for any $x_i \in U$, $i=1, \dots, k+1$. Again by lemma 1, no set of $k+1$ functions c_s is linear independent. Therefore, there exists a set of functions $f_1(x), \dots, f_k(x)$, such that for any s , $c_s(x) = \sum_{i=1}^k \beta_s^i f_i(x)$, i.e. $c_s \in \text{span}\{f_1, \dots, f_k\}$. Also, no less than k such functions exists, because otherwise, following $(3) \Rightarrow (2) \Rightarrow (1)$, we would have that $\text{rank}(\psi) < k$. Thus condition (3) holds. \square

Note that if ψ is symmetric and representation (2) holds, then g_i are linear combinations of f_i . Indeed, if we choose $x_j, j=1, \dots, k$, such that $\text{rank}(\{f_i(x_j)\}_{i=1, j=1}^{k,k}) = k$, then we can solve the system

$$\sum_{i=1}^k f_i(x_j) g_i(y) = \sum_{i=1}^k f_i(y) g_i(x_j), j = 1, \dots, k$$

to obtain $g_i(y) = \sum_{j=1}^k \gamma_{ij} f_j(y)$.

An immediate corollary of Theorem 1 is that for ψ as assumed there, which has a finite rank, $\text{rank}(\psi) = k$, the integral operator

$$L_\psi : f \mapsto \int_V \psi(x, y) f(y) dy, \quad f : U \rightarrow \mathbb{R},$$

is of finite rank.

Example 1 Consider the Euclidean metric in \mathbb{R}^n , $\psi(x, y) = \sum_{s=1}^n (x^s - y^s)^2$, where x^s are the components of $x \in \mathbb{R}^n$. Because of the global representation

$$\psi(x, y) = \sum_{s=1}^n (x^s)^2 * 1 - \underbrace{2x^1y^1 \dots - 2x^ny^n}_n + 1 * \sum_{s=1}^n (y^s)^2,$$

by Theorem 1, $\text{rank}(\psi) = n + 2$, i.e. the Euclidean metric has a finite rank of 2 more than the number of dimensions.

Definition 2 A collection of functions $f_s : U \rightarrow \mathbb{R}$ is said to be locally linear independent in U if for any open subset V of U we have that $\sum_s \alpha_s f_s(x) = 0$, for almost all $x \in V$ only if $\alpha_s = 0$ for all s .

Note that local linear independence is stronger than linear independence and the former implies the latter.

Lemma 2 If functions $f_1(x), \dots, f_k(x)$ are locally linearly independent in U and $U_i \subset U, i=1, \dots, k$ are open subsets, then there exist x_1, \dots, x_k , such that $x_i \in U_i$ and $\text{rank}(\{f_i(x_j)\}_{i=1, j=1}^{k, k}) = k$.

Proof. We can repeat the lines of the proof of lemma 1 with the only change at each selection step to choose $x_i \in U_i$.

Since $f_1(x)$ can not be identically zero in U_1 , there is $x_1 \in U_1$ such that $f_1(x_1) \neq 0$. Assuming that there are $x_i \in U_i, i=1, \dots, k-1$, such that $\det(A_1 := \{f_i(x_j)\}_{i=1, j=1}^{k-1, k-1}) \neq 0$, we can choose $x_k \in U_k$ such that for $x = x_k$

$$\det \begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots & f_k(x_1) \\ \dots & \dots & \dots & \dots \\ f_1(x_{k-1}) & f_2(x_{k-1}) & \dots & f_k(x_{k-1}) \\ f_1(x) & f_2(x) & \dots & f_k(x) \end{pmatrix} = \sum_{i=1}^{k-1} A_i f_i(x) \neq 0.$$

Otherwise $\sum_{i=1}^k A_i f_i(x) = 0, \forall x \in U_k$ and f_i would not be locally linear independent. \square

Example 2 For any k and distinct indices s_1, \dots, s_k , power functions $\{x^{s_l}\}_{l=1}^k$ are locally linear independent in any open $U \subset \mathbb{R}^n$. Indeed, it is sufficient

to show the claim for $n = 1$ and $U = (a - \delta, a + \delta)$, $\delta > 0$. Assume that $s_1 < s_2 < \dots < s_k$. Let $\sum_{l=1}^k \alpha_l (x + a)^{s_l} = 0$, $\forall x \in (-\delta, \delta)$. Then the $(s_1)^{th}, (s_2)^{th}, \dots, (s_k)^{th}$ derivatives of the left-hand-side sum at $x = 0$ have to be zeroes. Thus for any $m = 1, \dots, k$,

$$\sum_{l=m}^k \alpha_l \frac{s_l!}{(s_l - s_m)!} a^{s_l - s_m} = 0.$$

Which leads to the only possible choice $\alpha_l = 0$, for all l .

In fact, we have a stronger result.

Lemma 3 *The power functions $\{x^s\}_{[s] \geq 0}^\infty$ are locally linear independent in \mathbb{R}^n .*

Proof. Let $U \subset \mathbb{R}^n$ be an open set. Consider the Hilbert space $L_2(U)$ of square-integrable functions in U . We have that

$$L_2(U) \subset \overline{\text{span}\{x^s|_U, [s] \geq 0\}},$$

the closure of the span of the power functions (see [16], sec. 46). By the Gram-Schmidt method we can generate an orthonormal basis $\{\phi_s(x)\}_{[s] \geq 0}$ of $L_2(U)$ from $\{x^s|_U\}_{[s] \geq 0}$. Since $\{\phi_s(x)\}_{[s] \geq 0}$ can not be linear dependent, so can not be $\{x^s|_U\}_{[s] \geq 0}$. \square

For subsets $U_l, l=1, \dots, m$, of \mathbb{R}^n with $\bigotimes_{l=1}^m U_l$ we denote the product $U_1 \times \dots \times U_m$.

Definition 3 *Function $\psi : U \times V \rightarrow \mathbb{R}$ is said to have full rank almost everywhere (a.e.) in $U \times V$ if for any number $k \in \mathbb{N}$ and open sets $U_i \subset U$ and $V_j \subset V$, $i, j = 1, \dots, k$ we have that*

$$\text{rank}(\{\psi(x_i, y_j)\}_{i=1, j=1}^{k, k}) \leq k,$$

$\forall x_i \in U_i$ and $\forall y_j \in V_j$, with an equality for at least one choice of x_i 's and y_j 's.

Proposition 2 *Let $\psi \in C^\infty(U \times V)$ be an analytic function in V about a point $p \in V$ and for any $k \in \mathbb{N}$, there is a set of k Taylor functions $c_s(x)$, which are locally linear independent in U . Then ψ has full rank a.e. in $U \times V$.*

Proof. Suppose that there exist k and open sets $U_i \subset U$ and $V_j \subset V$, $i, j = 1, \dots, k$, such that for all $x_i \in U_i$ and $y_j \in V_j$, $\text{rank}(\{\psi(x_i, y_j)\}_{i,j=1}^{k,k}) < k$.

If we fix $x_i \in U_i$, $i = 1, \dots, k$, then by lemma 2, $\{\psi(x_i, y)\}_{i=1}^k$ are not locally linear independent functions of y in V . Consequently there exists an open set $W(x) \subset V$, $x = (x_1, \dots, x_k)$, where $\{\psi(x_i, y)|_{W(x)}\}_{i=1}^k$ are linearly dependent. Thus there are $\alpha_i(x)$, not all zero, such that

$$\sum_{i=1}^k \alpha_i(x) \psi(x_i, y) = 0, \forall y \in W(x).$$

If we expand $\psi(x_i, y)$'s into their Taylor series (assuming p is the origin), we obtain

$$\sum_s \sum_{i=1}^k \alpha_i(x) c_s(x_i) y^s = 0, \forall y \in W(x).$$

By local linear independence of the power functions y^s it follows that

$$\sum_{i=1}^k \alpha_i(x) c_s(x_i) = 0, \forall s.$$

Since the last condition is true for any $x_i \in U_i$, $i=1, \dots, k$, no k functions c_s would be locally linear indepent in U , which contradicts the theorem assumption. Therefore, ψ has full rank in any open subset of $\bigotimes_{l=1}^k U \bigotimes_{l=1}^k V$. \square

The following fact will be useful.

Lemma 4 *Any collection \mathcal{O} of (nonempty) open and disjoint subsets of \mathbb{R}^n is countable.*

Proof. Let $O \in \mathcal{O}$. Then there is at least one vector $q \in O$ with all rational coordinates. By applying the axiom of choice we define a map $O \mapsto (k_1, l_1, k_2, l_2, \dots, k_n, l_n)$, where k_i, l_i , $i = 1, \dots, n$, are whole numbers such that $q = (2^{k_1} l_1, \dots, 2^{k_n} l_n) \in O$. \mathcal{O} is countable since this map is injective one from \mathcal{O} to \mathbb{Q}^{2n} . \square

Let ν_n be the Lebesgue measure in \mathbb{R}^n . The next measure-theoretic result has important consequences.

Proposition 3 *If $f \in C^\infty(U)$ for an open subset U of \mathbb{R}^n , such that the set $V_0 := \{x \in U | f(x) = 0\}$ has no interior points, then $\nu_n(V_0) = 0$.*

Proof. Define $V_1 := \{x \in U \mid \frac{\partial f}{\partial x}|_x = 0\}$. When $n > 1$, $\frac{\partial f}{\partial x}$ is a gradient vector and $\frac{\partial f}{\partial x}|_x = 0$ means that all partial derivatives at point x vanish. Both V_0 and V_1 are closed in U . By the implicit function theorem, for any point $x \in V_0 \setminus V_1$ there is an open ball $B_x = B_x(r_x)$ of radius $r_x > 0$ and a hypersurface S_x of dimension $n-1$, the graph of a function of $n-1$ variables, such that $x \in S_x = B_x \cap V_0 \setminus V_1$. Therefore there exists a collection $\{S_\alpha\}_\alpha$ that partitions $V_0 \setminus V_1$, $V_0 \setminus V_1 = \cup_\alpha S_\alpha$, and such that S_α are disjoint and with measure zero, $\nu_n(S_\alpha) = 0$. Each S_α is obtained by merging countable number of hypersurfaces S_x . Indeed, let for every x , U_x be the support of the function which graph is S_x . We may assume that every U_x is a subset of one of the coordinate hyperplane $x^i = 0$, $i = 1, \dots, n$. Those of them that are subsets of $x^i = 0$ form a countable disjoint collection of open subsets, since any two U_x on $x^i = 0$ that intersect with each other can be seamlessly merged into the support of one function.

Observe that if $x \in S_\alpha$ and $y \in S_\beta$ for $\alpha \neq \beta$ then $B_x(r_x/2) \cap B_y(r_y/2) = \emptyset$, for otherwise either $x \in B_y(r_y)$ or $y \in B_x(r_x)$, a contradiction. Therefore for the open sets $U_\alpha := \cup_{x \in S_\alpha} B_x(r_x/2)$, we have $U_\alpha \cap V_0 \setminus V_1 = S_\alpha$ and U_α are disjoint. By the virtue of Lemma 4 the collection of open sets $\{U_\alpha\}_\alpha$ is countable and so is $\{S_\alpha\}_\alpha$. Therefore $\nu_n(V_0 \setminus V_1) = 0$ and $\nu_n(V_0) = \nu_n(V_1 \cap V_0)$.

We may repeat the same analysis for higher derivatives of order $l > 1$ and apply induction on l . Define

$$V_l := \{x \in U \mid \text{For all } s = (s_1, \dots, s_n), \text{ s.t. } [s] = l : \frac{\partial^s f}{\partial x^s}|_x = 0\}.$$

For example, let $l = 2$ and let

$$V_1^i := \{x \in U \mid \frac{\partial f}{\partial x^i}|_x = 0\}, \quad V_2^i := \{x \in U \mid \frac{\partial^2 f}{\partial x^i \partial x^j}|_x = 0, \text{ for all } j = 1, \dots, n\}.$$

The above argument for showing $\nu_n(V_0) = \nu_n(V_1 \cap V_0)$ is applicable to all pairs (V_1^i, V_2^i) and consequently $\nu_n(V_1^i) = \nu_n(V_2^i \cap V_1^i)$, $i = 1, \dots, n$. Next we use the fact that for any four measurable sets satisfying $B \subset A$, $\nu_n(B) = \nu_n(A)$, $D \subset C$ and $\nu_n(D) = \nu_n(C)$ we have $\nu_n(B \cap D) = \nu_n(A \cap C)$, to conclude that $\nu_n(\cap_{i=1}^n V_1^i) = \nu_n(\cap_{i=1}^n V_1^i \cap \cap_{i=1}^n V_2^i)$ or equivalently $\nu_n(V_1) = \nu_n(V_1 \cap V_2)$.

By induction, $\nu_n(V_{l-1} \setminus V_l) = 0$ for any $l > 0$. Therefore given that $\nu_n(V_0) = \nu_n(\cap_{m=0}^{l-1} V_m)$, from

$$\nu_n(\cap_{m=0}^{l-1} V_m) = \nu_n((\cap_{m=0}^{l-1} V_m) \setminus V_l) + \nu_n(\cap_{m=0}^l V_m) = \nu_n(\cap_{m=0}^l V_m)$$

we obtain $\nu_n(V_0) = \nu_n(\cap_{m=0}^l V_m)$. Finally, we realize that the set $\cap_{s=1}^\infty V_s$ should be empty, for otherwise, with all vanishing derivatives at one point, the function f would vanish in an open subset of U . Thus we conclude that $\nu_n(V_0) = \nu_n(\cap_{l=1}^\infty V_l) = 0$. \square

By applying the above proposition for the function $\det(\{\psi(x_i, y_j)\}_{i,j=1}^{k,k})$ we immediately obtain the following result.

Corollary 1 *Under the conditions of Theorem 2, for any $k \in \mathbb{N}$*

$$\nu_{2kn}(D_k := \{(x_1, \dots, x_k, y_1, \dots, y_k) \in \bigotimes_{l=1}^k U \bigotimes_{l=1}^k V : \text{rank}(\{\psi(x_i, y_j)\}_{i,j=1}^{k,k}) < k\}) = 0,$$

which explains "a.e." term in the full rank definition.

Next we show that Theorem 2 can be applied to a large class of symmetric analytic functions. With $x.y = \sum_{l=1}^n x^l y^l$ we will denote the dot product in \mathbb{R}^n .

Corollary 2 *Let $\psi(x, y) = h(x.y)$, $(x, y) \in U \times V$ and for $W = \{x.y | x \in U, y \in V\} \subset \mathbb{R}$, $0 \in W$. Let also h be analytic function in W about the origin 0 and such that $h^{(s)}(0) \neq 0$ for infinitely many s . Then ψ has full rank a.e. in $U \times V$.*

Proof. Observe that

$$c_s(x) = \frac{\partial^s}{\partial y^s} \psi(x, y)|_{y=0} = x^s h^{(s)}(0).$$

Since power functions are locally linear independent, for any k there exist k functions c_s that are locally linear independent in U . Thus the conditions of Theorem 2 are fulfilled. \square

Example 3 *Consider the distance on the n -sphere \mathbb{S}^n given by*

$$\psi(x, y) = \cos^{-1}(x.y), x, y \in \mathbb{S}^n.$$

Conditions of corollary 2 are met for the function $h(z) = \cos^{-1}(z)$, $z \in [-1, 1]$. Indeed h is analytic about 0 with infinitely many Taylor coefficients non-zero

$$\cos^{-1}(z) = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.$$

Example 3 shows that the distance on the n-sphere is of full rank almost everywhere, a fact of importance to some inverse problems on the sphere.

Remark 2 *The full rank property of $\cos^{-1}(x.y)$, $x, y \in \mathbb{S}^n$ follows from another standard result in linear operator theory. In the Hilbert space $L_2(\mathbb{S}^n)$ of square integrable functions on \mathbb{S}^n , the operator with kernel $\cos^{-1}(x.y)$ is symmetric and thus we have the representation $\cos^{-1}(x.y) = \sum_{k \geq 1} \lambda_k \phi_k(x) \phi_k(y)$, where λ_k and ϕ_k are the eigenvalues and eigenvectors of this operator (see Theorems of Hilbert and Schmidt, [16], sec. 97). Since ϕ_k 's form orthonormal system in $L_2(\mathbb{S}^n)$, they are necessarily linear independent. In fact, one can expect that they are locally linear independent, which will give us the full rank result. Our approach, however, is stronger in a sense. For example, according to corollary 2, $\cos(x.y)$, $x, y \in \mathbb{R}^n$, has full rank a.e., but one can not use Hilbert space argument to show it, since $\cos(x.y)$ is not square integrable in $\mathbb{R}^n \times \mathbb{R}^n$.*

3 On solvability of some systems of matrix equations

Let U and V be open subsets of \mathbb{R}^n and $\eta : U \times U \rightarrow \mathbb{R}^n$ be a continuous vector valued function. Let also $\mathcal{P} = \{p_i \in U\}_{i=1}^k$ be a set of points in U . For any discrete function f on \mathcal{P} , $f = (f_i = f(x_i))_{i=1}^k \in \mathbb{R}^k$, define $\Sigma[f] = \{\Sigma[f]_j\}_{j=1}^k$,

$$\Sigma[f]_j := \sum_{i=1}^k f_i \eta(x_j, x_i) \eta(x_j, x_i)^T, \quad j = 1, \dots, k.$$

We have $\Sigma[f] \in [Sym_n]^k$, where Sym_n is the set of symmetric $n \times n$ matrices. Note that $K(x, y) := \eta(x, y) \eta(x, y)^T$ is always symmetric and positive semi-definite and so K is Mercer kernel (see [5]).

If $C = \Sigma[f^0]$ for $f^0 \in \mathbb{R}^n$, then f^0 will be a solution of the system

$$\sum_{i=1}^k f_i Y_{ji} = C_j, \quad j = 1 \dots k, \quad (7)$$

where $Y_{ji} = \eta(x_j, x_i) \eta(x_j, x_i)^T \in Sym_n$. We say that f^0 can be recovered if system (7) has a unique solution. System (7) is in fact a discretization of the integral tensor equation (3).

We re-arrange the components of each of the k^2 $n \times n$ -matrices $Y_{ji} = \{Y_{ji}^{lm}\}_{l=1, m=1}^{n, n}$, $i, j=1, \dots, k$, into new $k \times k$ -matrices $B_{lm} = \{Y_{ji}^{lm}\}_{i=1, j=1}^{k, k} \in \mathbb{R}^{k \times k}$, $l, m=1, \dots, n$. Then we order all n^2 matrices B_{lm} vertically to obtain a $n^2 k \times k$ -matrix

$$\mathcal{Y} = [B_{11}^T \dots B_{1n}^T \dots B_{n1}^T \dots B_{nn}^T]^T \in \mathbb{R}^{n^2 k \times k}.$$

Similarly, matrices C_j can be unfolded in a large $n^2 k$ -vector

$$\mathcal{C} = [C_1^{11} \dots C_k^{11} C_1^{12} \dots C_k^{12} \dots C_1^{nn} \dots C_k^{nn}]^T,$$

and equation (7) can be written as

$$\mathcal{Y}f = \mathcal{C}, f \in \mathbb{R}^n.$$

Occasionally we will write $\mathcal{Y}(\mathcal{P})$ to emphasize that \mathcal{Y} is contingent on the choice of \mathcal{P} .

In fact, when $C = \Sigma[f^0]$, f^0 can be recovered if and only if

$$\sum_{i=1}^k f_i \text{tr}(Y_{ji}) = \text{tr}(C_j), \quad j = 1 \dots k, \quad (8)$$

has a unique solution (f^0) . Define

$$\psi(x, y) := \text{tr}(\eta(x, y)\eta(x, y)^T) = \|\eta(x, y)\|^2$$

and let $\Psi = \{\psi(x_i, y_j)\}_{i, j=1}^{k, k}$ and $c = (\text{tr}(C_1), \dots, \text{tr}(C_k))^T$. Then (8) can be written as $\Psi f = c$. The rank of function ψ , which we studied in the previous section, determines the solvability of (8).

Since $C_j = \Sigma[f^0]_j \in \text{span}(Y_{j1}, \dots, Y_{jk})$, we have that

$$\text{rank}(\mathcal{Y}|\mathcal{C}) = \text{rank}(\mathcal{Y}),$$

where $\mathcal{Y}|\mathcal{C}$ is matrix \mathcal{Y} with vector \mathcal{C} attached as a last column. The system of linear equations (7) has a unique solution if and only if \mathcal{Y} is of full rank, $\text{rank}(\mathcal{Y}) = k$.

Therefore it is of importance to find under what circumstances \mathcal{Y} has full rank. If the points of \mathcal{P} are chosen by a continuous distribution of M , what will be the probability for \mathcal{Y} to be of smaller rank?

The properties of \mathcal{Y} are contingent on the choice of the vector function η . It is natural first to consider the simplest choice $\eta(x, y) = x - y$, corresponding

to the Euclidean distance vector function. Observe that in this case $\psi(x, y) = \text{tr}((y - x)(y - x)^T) = \|y - x\|^2$ and as we already have found in the previous section (Example 1), $\text{rank}(\psi) = n + 2$. Thus

$$\text{rank}(\Psi) \leq n + 2.$$

In the light of this fact, we may expect that the rank of \mathcal{Y} is also bounded for this choice of η .

Proposition 4 *For the map $\eta(x, y) = x - y$ in \mathbb{R}^n ,*

$$\text{rank}(\mathcal{Y}(\mathcal{P})) = \min\{k, \frac{(n+1)(n+2)}{2}\},$$

and therefore, (7) has not a unique solution for $k > \frac{(n+1)(n+2)}{2}$.

Proof. With respect to a fixed coordinate system in \mathbb{R}^n , we identify the points $p_i \in \mathcal{P}$ with vectors $v_i = (v_i^1, \dots, v_i^n) \in \mathbb{R}^n$. Then the matrices Y_{ji} have the following structure

$$Y_{ji} = \begin{pmatrix} (v_i^1 - v_j^1)^2 & (v_i^1 - v_j^1)(v_i^2 - v_j^2) & \dots & (v_i^1 - v_j^1)(v_i^n - v_j^n) \\ (v_i^2 - v_j^2)(v_i^1 - v_j^1) & (v_i^1 - v_j^1)^2 & \dots & (v_i^2 - v_j^2)(v_i^n - v_j^n) \\ \dots & \dots & \dots & \dots \\ (v_i^n - v_j^n)(v_i^1 - v_j^1) & (v_i^n - v_j^n)(v_i^2 - v_j^2) & \dots & (v_i^n - v_j^n)^2 \end{pmatrix} \quad (9)$$

First we look at the case $n = 1$, because is much simpler and gives us intuition needed for the general case. We will show that for any eight real numbers $a, b, c, d, a_1, b_1, c_1, d_1$, the matrix

$$\mathbf{X} = \begin{pmatrix} (a - a_1)^2 & (b - a_1)^2 & (c - a_1)^2 & (d - a_1)^2 \\ (a - b_1)^2 & (b - b_1)^2 & (c - b_1)^2 & (d - b_1)^2 \\ (a - c_1)^2 & (b - c_1)^2 & (c - c_1)^2 & (d - c_1)^2 \\ (a - d_1)^2 & (b - d_1)^2 & (c - d_1)^2 & (d - d_1)^2 \end{pmatrix} \quad (10)$$

is singular with $\text{rank}(\mathbf{X}) \leq 3$. Since any 4×4 sub-matrix of \mathcal{Y} has the structure of \mathbf{X} , it will follow that $\text{rank}(\mathcal{Y}) \leq 3$.

For any $k \times k$ matrix Z let $l^j(Z)$ denote the j -th column vector of Z and $\mathcal{L}(Z) = \text{span}\{l^1(Z), \dots, l^k(Z)\}$, the linear space defined by the column vectors of Z .

To see why $\text{rank}(\mathbf{X}) \leq 3$, we define $\underline{1} = (1, 1, 1, 1)^T$, $\underline{x} = (a_1, b_1, c_1, d_1)^T$, $\underline{x}^2 = (a_1^2, b_1^2, c_1^2, d_1^2)^T$, and consider following expression for the columns of X

$$l^1(X) = a^2 \underline{1} - 2a\underline{x} + \underline{x}^2, \quad l^2(X) = b^2 \underline{1} - 2b\underline{x} + \underline{x}^2$$

$$l^3(X) = c^2 \underline{1} - 2c\underline{x} + \underline{x}^2, \quad l^4(X) = d^2 \underline{1} - 2d\underline{x} + \underline{x}^2.$$

Then we observe that $\mathcal{L}(X) \subset \mathcal{X} := \text{span}\{\underline{1}, \underline{x}, \underline{x}^2\}$, and therefore $\text{rank}(X) \leq \dim(\mathcal{X}) \leq 3$.

Let check now the general case. The $n^2 k \times k$ matrix \mathcal{Y} looks like this

$$\mathcal{Y} = \begin{pmatrix} (v_1^1 - v_1^1)(v_1^1 - v_1^1) & (v_1^1 - v_2^1)(v_1^1 - v_2^1) & \dots & (v_1^1 - v_k^1)(v_1^1 - v_k^1) \\ (v_2^1 - v_1^1)(v_2^1 - v_1^1) & (v_2^1 - v_2^1)(v_2^1 - v_2^1) & \dots & (v_2^1 - v_k^1)(v_2^1 - v_k^1) \\ \dots & \dots & \dots & \dots \\ (v_k^1 - v_1^1)(v_k^1 - v_1^1) & (v_k^1 - v_2^1)(v_k^1 - v_2^1) & \dots & (v_k^1 - v_k^1)(v_k^1 - v_k^1) \\ (v_1^1 - v_1^1)(v_1^2 - v_1^2) & (v_1^1 - v_2^1)(v_1^2 - v_2^2) & \dots & (v_1^1 - v_k^1)(v_1^2 - v_k^2) \\ (v_2^1 - v_1^1)(v_2^2 - v_2^2) & (v_2^1 - v_2^1)(v_2^2 - v_2^2) & \dots & (v_2^1 - v_k^1)(v_2^2 - v_k^2) \\ \dots & \dots & \dots & \dots \\ (v_k^1 - v_1^1)(v_k^2 - v_k^2) & (v_k^1 - v_2^1)(v_k^2 - v_k^2) & \dots & (v_k^1 - v_k^1)(v_k^2 - v_k^2) \\ \dots & \dots & \dots & \dots \\ (v_1^1 - v_1^1)(v_1^n - v_1^n) & (v_1^1 - v_2^1)(v_1^n - v_2^n) & \dots & (v_1^1 - v_k^1)(v_1^n - v_k^n) \\ (v_2^1 - v_1^1)(v_2^n - v_2^n) & (v_2^1 - v_2^1)(v_2^n - v_2^n) & \dots & (v_2^1 - v_k^1)(v_2^n - v_k^n) \\ \dots & \dots & \dots & \dots \\ (v_k^1 - v_1^1)(v_k^n - v_k^n) & (v_k^1 - v_2^1)(v_k^n - v_2^n) & \dots & (v_k^1 - v_k^1)(v_k^n - v_k^n) \\ \dots & \dots & \dots & \dots \\ (v_1^n - v_1^1)(v_1^1 - v_1^1) & (v_1^n - v_2^1)(v_1^1 - v_2^1) & \dots & (v_1^n - v_k^1)(v_1^1 - v_k^1) \\ (v_2^n - v_1^1)(v_2^1 - v_2^1) & (v_2^n - v_2^1)(v_2^1 - v_2^1) & \dots & (v_2^n - v_k^1)(v_2^1 - v_k^1) \\ \dots & \dots & \dots & \dots \\ (v_k^n - v_1^1)(v_k^1 - v_k^1) & (v_k^n - v_2^1)(v_k^1 - v_k^1) & \dots & (v_k^n - v_k^1)(v_k^1 - v_k^1) \\ \dots & \dots & \dots & \dots \\ (v_1^n - v_1^1)(v_1^n - v_1^n) & (v_1^n - v_2^n)(v_1^n - v_2^n) & \dots & (v_1^n - v_k^n)(v_1^n - v_k^n) \\ (v_2^n - v_1^n)(v_2^n - v_2^n) & (v_2^n - v_2^n)(v_2^n - v_2^n) & \dots & (v_2^n - v_k^n)(v_2^n - v_k^n) \\ \dots & \dots & \dots & \dots \\ (v_k^n - v_1^n)(v_k^n - v_k^n) & (v_k^n - v_2^n)(v_k^n - v_k^n) & \dots & (v_k^n - v_k^n)(v_k^n - v_k^n) \end{pmatrix}$$

Consider the s -th column of \mathcal{Y} . Its elements are formed by multiplication of two differences, which we will express formally as differences of sub-columns (1) – (2) and (3) – (4). Next we show the four sub-columns of

s -th column as rows

$$\begin{aligned}
(1)^T &: \underbrace{v_1^1, \dots, v_k^1}_{k} \dots \underbrace{v_1^1, \dots, v_k^1}_{k} \underbrace{v_1^2, \dots, v_k^2}_{k} \dots \underbrace{v_1^2, \dots, v_k^2}_{k} \dots \underbrace{v_1^n, \dots, v_k^n}_{k} \dots \underbrace{v_1^n, \dots, v_k^n}_{k} \\
&\quad \underbrace{\hspace{1.5cm}}_n \quad \underbrace{\hspace{1.5cm}}_n \quad \underbrace{\hspace{1.5cm}}_n \\
(2)^T &: \underbrace{v_s^1, \dots, v_s^1}_{k} \dots \underbrace{v_s^1, \dots, v_s^1}_{k} \underbrace{v_s^2, \dots, v_s^2}_{k} \dots \underbrace{v_s^2, \dots, v_s^2}_{k} \dots \underbrace{v_s^n, \dots, v_s^n}_{k} \dots \underbrace{v_s^n, \dots, v_s^n}_{k} \\
&\quad \underbrace{\hspace{1.5cm}}_n \quad \underbrace{\hspace{1.5cm}}_n \quad \underbrace{\hspace{1.5cm}}_n \\
(3)^T &: \underbrace{v_1^1, \dots, v_k^1}_{k} \dots \underbrace{v_1^n, \dots, v_k^n}_{k} \underbrace{v_1^1, \dots, v_k^1}_{k} \dots \underbrace{v_1^n, \dots, v_k^n}_{k} \dots \underbrace{v_1^1, \dots, v_k^1}_{k} \dots \underbrace{v_1^n, \dots, v_k^n}_{k} \\
(4)^T &: \underbrace{v_s^1, \dots, v_s^1}_{k} \dots \underbrace{v_s^n, \dots, v_s^n}_{k} \underbrace{v_s^1, \dots, v_s^1}_{k} \dots \underbrace{v_s^n, \dots, v_s^n}_{k} \dots \underbrace{v_s^1, \dots, v_s^1}_{k} \dots \underbrace{v_s^n, \dots, v_s^n}_{k}
\end{aligned}$$

Formally, s -th column equals $((1) - (2))((3) - (4))$ if the operations are taken component-wise.

For a vector x , with $x \hookrightarrow_m$ we denote the vector obtained from x by shifting m positions in the right. For example, $(0, 1, 0, 0) \hookrightarrow_2 = (0, 0, 0, 1)$. Let

$$\underline{x}_1 = (\underbrace{v_1^1, \dots, v_k^1, v_1^2, \dots, v_k^2, \dots, v_1^n, \dots, v_k^n}_{nk}, \underbrace{0, \dots, 0}_{(n-1)nk})^T \in \mathbb{R}^{n^2k},$$

$$\underline{x}_j = \underline{x}_1 \hookrightarrow_{(j-1)nk}, \text{ for } j=1, \dots, n,$$

$$\underline{z}_1 = (v_1^1, \dots, v_k^1 \underbrace{0, \dots, 0}_{(n-1)k}, v_1^2, \dots, v_k^2 \underbrace{0, \dots, 0}_{(n-1)k}, \dots, v_1^n, \dots, v_k^n \underbrace{0, \dots, 0}_{(n-1)k})^T \in \mathbb{R}^{n^2k},$$

$$\underline{z}_j = \underline{z}_1 \hookrightarrow_{(j-1)k}, \text{ for } j=1, \dots, n,$$

$$\underline{1}_1 = (\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{(n-1)k}, \underbrace{0, \dots, 0}_{(n-1)nk})^T \in \mathbb{R}^{n^2k},$$

$$\underline{1}_j = \underline{1}_1 \hookrightarrow_{(j-1)}, \text{ for } j=1, \dots, n, \text{ and}$$

$$\underline{1}_{ij} = (\underline{1}_1 \hookrightarrow_{(i-1)nk}) + (\underline{1}_1 \hookrightarrow_{(i-1)nk+(j-1)k}), \text{ for } j = 1, \dots, n, i < j.$$

Let $l^0 \in \mathbb{R}^{n^2k}$ be the vector

$$(v_1^1 v_1^1, v_2^1 v_2^1, \dots, v_k^1 v_k^1, v_1^1 v_1^2, v_2^1 v_2^2, \dots, v_k^1 v_k^2, \dots, v_1^1 v_1^n, v_2^1 v_2^n, \dots, v_k^1 v_k^n,$$

$$\begin{aligned} &v_1^2 v_1^1, v_2^2 v_2^1, \dots, v_k^2 v_k^1, v_1^2 v_1^2, v_2^2 v_2^2, \dots, v_k^2 v_k^2, \dots, v_1^2 v_1^n, v_2^2 v_2^n, \dots, v_k^2 v_k^n, \dots \\ &v_1^n v_1^1, v_2^n v_2^1, \dots, v_k^n v_k^1, v_1^n v_1^2, v_2^n v_2^2, \dots, v_k^n v_k^2, \dots, v_1^n v_1^n, v_2^n v_2^n, \dots, v_k^n v_k^n)^T. \end{aligned}$$

Then we can express s -th column of \mathcal{Y} as

$$l^s(\mathcal{Y}) = l^0 - \sum_{j=1}^n v_s^j(\underline{x}_j + \underline{z}_j) + \sum_{j=1}^n (v_s^j)^2 \underline{1}_j + 2 \sum_{i < j} v_s^i v_s^j \underline{1}_{ij}.$$

Note that $\underline{1}_j = \underline{1}_{1j}$ and therefore

$$\mathcal{L}(\mathcal{Y}) \subset \text{span}\{l^0, \underline{1}_{ij}, (\underline{x}_j + \underline{z}_j)\}_{j=1, i < j}^n.$$

Finally we obtain

$$\dim(\mathcal{L}(\mathcal{Y})) \leq 1 + \frac{n(n+1)}{2} + n = \frac{(n+1)(n+2)}{2}.$$

□

Another matrix of potential interest is obtained by arranging matrices $Y_{ji}, i, j=1, \dots, k$ in one $kn \times kn$ matrix

$$\mathcal{Z} = \begin{pmatrix} Y_{11} & Y_{21} & \dots & Y_{k1} \\ Y_{12} & Y_{22} & \dots & Y_{k2} \\ \dots & \dots & \dots & \dots \\ Y_{1k} & Y_{2k} & \dots & Y_{kk} \end{pmatrix} \quad (11)$$

What is the rank of \mathcal{Z} for the choice $\eta(x, y) = y - x$?

Proposition 5 *For the map $\eta(x, y) = x - y$ in \mathbb{R}^n ,*

$$\text{rank}(\mathcal{Z}) = \min\{k, n(n+2)\}.$$

Proof. We will find a representation of $(s-1)n + m$ -th column of \mathcal{Z} for $s=1, \dots, k$ and $m=1, \dots, n$, as a linear combination of $n(n+2)$ basis column vectors and that will proof the claim.

The elements of $(s-1)n + m$ -th column of \mathcal{Y} are formed by multiplication of two differences, which we express formally as differences of sub-columns (1) – (2) and (3) – (4). We show these four sub-columns as rows

$$(1)^T : \underbrace{\underbrace{v_s^1, \dots, v_s^n}_n \underbrace{v_s^1, \dots, v_s^n}_n \dots \underbrace{v_s^1, \dots, v_s^n}_n}_{k}, \quad (2)^T : \underbrace{v_1^1, \dots, v_1^n}_n \underbrace{v_2^1, \dots, v_2^n}_n \dots \underbrace{v_k^1, \dots, v_k^n}_n$$

$$(3)^T : \underbrace{\underbrace{v_s^l, \dots, v_s^l}_n \underbrace{v_s^l, \dots, v_s^l}_n \dots \underbrace{v_s^l, \dots, v_s^l}_n}_{k}, \quad (4)^T : \underbrace{v_1^l, \dots, v_1^l}_n \underbrace{v_2^l, \dots, v_2^l}_n \dots \underbrace{v_k^l, \dots, v_k^l}_n$$

Then, $(s-1)n+m$ -th column equals $((1)-(2))((3)-(4))$ if the operations are taken component-wise. Let

$$\underline{x}_1^m = (\underbrace{v_1^m, 0 \dots 0}_n, \underbrace{v_2^m, 0 \dots 0}_n, \dots, \underbrace{v_k^m, 0 \dots 0}_n)^T \in \mathbb{R}^{nk},$$

$$\underline{x}_j^m = \underline{x}_1^m \hookrightarrow_{(j-1)}, \text{ for } j=1, \dots, n,$$

$$\underline{w} = (\underbrace{v_1^1, \dots, v_1^n}_n \underbrace{v_2^1, \dots, v_2^n}_n, \dots, \underbrace{v_k^1, \dots, v_k^n}_n)^T \in \mathbb{R}^{nk},$$

$$\underline{1}_1 = (\underbrace{1, 0 \dots 0}_n, \underbrace{1, 0 \dots 0}_n, \dots, \underbrace{1, 0 \dots 0}_n)^T \in \mathbb{R}^{nk},$$

$$\underline{1}_j = \underline{1}_1 \hookrightarrow_{(j-1)}, \text{ for } j=1, \dots, n, \text{ and}$$

$$\underline{l}^m = (v_1^1 v_1^m, \dots, v_1^n v_1^m, v_2^1 v_2^m, \dots, v_2^n v_2^m, \dots, v_k^1 v_k^m, \dots, v_k^n v_k^m)^T \in \mathbb{R}^{nk}$$

Now we can write $(s-1)n+m$ -th column of \mathcal{Z} as

$$l^{(s-1)n+m}(\mathcal{Z}) = \underline{l}^m + \sum_{j=1}^n v_s^j \underline{x}_j^m - v_s^m \underline{w} - v_s^m \sum_{j=1}^n v_s^j \underline{1}_j.$$

Observe that $\underline{w} = \sum_{j=1}^n \underline{x}_j^j$. Therefore

$$\mathcal{L}(\mathcal{Z}) \subset \text{span}\{\underline{l}^m, \underline{1}_j, \underline{x}_j^m\}_{j=1, m=1}^{n, n},$$

and

$$\dim(\mathcal{L}(\mathcal{Z})) \leq n + n + n^2 = n(n+2).$$

□

In conclusion, the rank of matrices \mathcal{Y} and \mathcal{Z} depend on the rank of the bi-variate function ψ . If ψ has a finite rank then the ranks of \mathcal{Y} and \mathcal{Z} will be bounded as is the case with the Euclidean distance function $\psi(x, y) = \|x - y\|^2$. As a consequence, a distribution function f in \mathbb{R}^n can not be recovered from its covariance field $\Sigma[f]$.

Recalling Corollary 1 we immediately obtain the following

Corollary 3 *If $\psi(x, y) = \text{tr}(\eta(x, y)\eta(x, y)^T)$ has full rank a.e. in U and \mathcal{P} is a finite sample of points drawn by a continuous distribution on U , then $\mathcal{Y}(\mathcal{P})$ has full rank with probability one.*

According to Example 3, the square sphere distance, $\psi(x, y) = (\cos^{-1}(x \cdot y))^2$, is of full rank and Corollary 3 holds. Thus, in contrast to the Euclidean space, a function f on \mathbb{S}^n can always be recovered from the respected field $\Sigma[f]$.

4 Final discussion and simulation results

Here we briefly discuss the problem of regularization of ill-posed integral equations of type (4). We re-formulate it using the geodesic distance d on a Riemannian manifold with volume form V

$$\int_U d^2(p, q) f(q) dV(q) = g(p), \quad U \subset M.$$

Discretizing the above equation on a set $\{p_i\}_i$ of k points results to a system of linear equations $D\tilde{f} = \tilde{g}$, where $D = \{d^2(p_i, p_j)\}_{i,j=1}^{k,k}$. Although for a full-rank metric d on M , D will be almost always non-singular, its condition number, $\text{cond}(D) = \|D\| \|D^{-1}\|$, may increase dramatically as k increases, making solutions unstable.

Regularization of ill-posed linear problems is an area of active research (see [10], [17], [5] and [4]). Along with the classical Tikhonov regularization several new approaches are currently investigated. A good overview of the topic can be found in [2].

Instead of trying to regularize an ill-conditioned kernel $\psi = d^2$, we replace it with a well-conditioned element from a family of similar to it kernels. The procedure is applicable to all non-negative kernels.

Consider the class of kernels $\{(d(p, q) - \alpha)^2\}_{\alpha > 0}$. What would be a good element of this class that gives low (in average) conditional number for D ? Assuming U to be compact, we choose α that minimizes the expectation of $(d(X, Y) - \alpha)^2$ for $X, Y \sim \text{Unif}(U)$, and X and Y independent. It is a well known statistical fact that the optimal choice in this case is the expectation of $d(X, Y)$.

For example, on the unit sphere \mathbb{S}^n if we take U to be the whole manifold, then $E(d(X, Y)) = \pi/2$ and therefore, $\alpha = \pi/2$ is the optimal choice for it minimizes $E(d(X, Y) - \alpha)^2$. Next we show some empirical evidence confirming the efficiency of this choice.

For a series of values of k ranging from 50 to 900 we draw 20, k -samples on \mathbb{S}^2 , calculate their conditional numbers, and then report the mean conditional number for these 20 samples. The next table shows how the mean conditional number varies as k increases for the baseline choice $\alpha = 0$ and the alternative $\alpha = \pi/2$.

α	k=50	k=100	k=150	k=200	k=250	k=300	
0	0.0000	0.0002	0.0226	0.3745	3.4871s	1.1612	$\times 10^{+15}$
$\pi/2$	0.1291	0.0710	0.2865	4.1476	0.5755	0.1442	$\times 10^{+4}$
	k=400	k=500	k=600	k=700	k=800	k=900	
0	0.0264	0.0694	0.0562	0.5102	1.5067	4.5686	$\times 10^{+19}$
$\pi/2$	3.606	9.977	0.757	4.525	8.257	10.094	$\times 10^{+4}$

The conditional numbers for $\alpha = 0$ increase exponentially with the increase of k . In fact, for values $k > 250$ the determinant of D has an infinitesimal magnitude of e^{-196} and the double-precision float arithmetic is not enough for consistent calculations. On the other hand, the conditional number for $\alpha = \pi/2$ is rather stable and allows computations for k well above 1000. In conclusion, the simulation results are in good support of the choice $\alpha = Ed(X, Y)$, $X, Y \sim Unif(U)$.

The above described procedure can be used to improve the conditioning of a covariance operator. A way to achieve this is instead of definition (5), to use the following modified operator

$$G\Sigma(p) := \int_{\mathcal{U}(p)} G(p)(Exp_p^{-1}q)(Exp_p^{-1}q)^T [1 - \frac{\alpha(p)}{d(p, q)}]^2 f(p) dV(p), \quad (12)$$

assuming that all $\alpha(p) := Ed^2(p, X)$, $X \sim Unif(\mathcal{U}(p))$ are well defined. Theoretical and experimental analysis of (12), however, is out of the scope of this paper.

References

- [1] Balov, N. (2008) Covariance fields. arXiv:0807.4690v1 (submitted).
- [2] Cavalier, L., Nonparametric statistical inverse problems. 2008, IOP Publishing, Inverse Problems, 24.
- [3] Cavalier, L., Inverse problems with non-compact operators. 2006, Jour. Stat. Plan. and Infer., 136, 390-400.

- [4] Cavalier, L., and Tsybakov, A., Sharp adaptation for inverse problems with random noise. 2002, *Probab. Theory Related Fields*, 123, 323-354.
- [5] Cucker, F., and Smale, S., Best Choices for Regularization Parameters in Learning Theory: On the Bias-Variance Problem. 2002, *Found. Comput. Math.*, 2:413-428.
- [6] Cucker, F., and Smale, S., On the mathematical foundations of learning. 2002, *Bull. Amer. Math. Soc.*, 39, 1-49.
- [7] Golub, G., Loan, C. (1996) *Matrix Computations*. 1996, 3rd edition, Johns Hopkins.
- [8] Halmos, P., R., *A Hilbert Space Problem Book*. 1967, Van Nostrand, Princeton, N.Y.
- [9] Hoffmann, M., Nonlinear estimation for linear inverse problems with error in the operator. 2007, *Ann. Stat.* at press
- [10] Ivanov, V., *The Theory of Approximate Methods and Their Applications to the Numerical Solutions of Singular Integral Equations*. 1976, Nordhoff Int..
- [11] Kim, P., and Koo, Ja-Yong, Statistical Inverse Problems on Manifolds. 2005, *The Journal of Fourier Analysis and Applications*.
- [12] Koo, J.-Y., Optimal rates of convergence for nonparametric statistical inverse problems. 1993, *Ann. Statist.*, 21, 590-599.
- [13] Mair, B., Ruymgaart, F., Statistical estimation in Hilbert scale. 1996, *SIAM J. Appl. Math.*, 56, 1424-1444.
- [14] Ruymgaart, F., A unified approach to inversion problems in statistics. 1993, *Math. Methods Statistics*, 2, 130-146.
- [15] Ruymgaart, F., A short introduction to inverse statistical inference Lecture in IHP (Paris). 2001.
- [16] Riesz, F., and B. Sz.-Nagy, *Functional Analysis*. 1955, Ungar, New York.

- [17] Tikhonov, A., and Arsenin, V., Solutions of Ill-Posed Problems. 1977, W. H. Winston.

Department of Statistics,
Florida State University,
117 N. Woodward Ave.,
Tallahassee, Florida, 32306-4330
`balov@stat.fsu.edu`